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Creative Component

Math 599

Voting Methods: Another Way to Choose

Introduction

How many ways are there to vote? You circle one girl for Homecoming Queen and you circle one boy for Homecoming King. In the voting booth, you fill in one oval or punch out one chad for president. If you select more than one; your ballot is thrown away and your vote does not count. Most of the general public in American society only knows one voting method, the plurality method. It is used for public elections, school elections, and even family decision making. Is it the best method? Are there other methods that would make for a happier public, student body, or family choice?

This paper will focus on six different voting methods: Plurality Method, Borda Count, Runoff Method, Sequential Runoff Method, Approval Voting, and a pairwise voting method with a Condorcet Winner. Plurality method is the most common method in use. The voter is only allowed to vote for 1 candidate, and the one with the most votes wins. Borda, runoff and sequential runoff allow the voter to rank all candidates in order from highest to lowest. The Borda winner is found by assigning points to each candidate's ranking, while both runoff methods narrow the choices down to two candidates. The approval method allows everyone to vote for as many candidates as they wish; no ranking order. Then the votes are totaled and the candidate with the most votes wins. The Condorcet winner is decided by comparing each choice against each other. The candidate with a majority over every other candidate is the winner.

These methods are relevant to young adults, because they can be implemented into student's current and future lives. "Making a "correct decision" is an important concern because decision procedures affect all aspects of our lives" (Saari, 2001, p. 4). During the learning of this topic and after, students will be able to use their knowledge in real-life situations. Student elections are held in all secondary settings for student government, dance royalty, extracurricular offices, awards, or many other student body choices. Students could also use them in their own families to help decide on “little decisions” such as pizza orders or movie rentals. Then students can move on to “bigger decisions” such as class trip plans, school or local elections.

This topic has a role in the secondary curriculum for several reasons. First, it will show competence in the NCTM Grades 9-12 Standards of Problem Solving, Reasoning & Proof, Communication, and Connections. Analyzing voting methods will help student’s decision making and problem solving techniques. The use of these different methods will make a student consider open-ended problems and make choices that he/she can justify. “High school students should be good critics, and good self-critics. They should be able to generate explanations, formulate questions, and write arguments that teachers, coworkers, or mathematicians would consider logically correct and coherent” (National Council of Teachers of Mathematics [NCTM], 2000, p. 349). Another reason is that as a student progresses through their learning, understanding, and implementation of the six voting methods, the six levels of Bloom’s cognitive complexity; Knowledge, Comprehension, Application, Analysis, Synthesis, and Evaluation are also achieved (Slavin, 1997). The final reason is that the Minnesota Department of Education has included the comparison of these voting methods as a benchmark in the Data Analysis Strand (Minnesota Department of Education [MDE], 2005). Starting in the spring of 2006, all 11th grade students will be tested on this information as part of the Minnesota Comprehensive Assessments Series II (MDE, 2005).

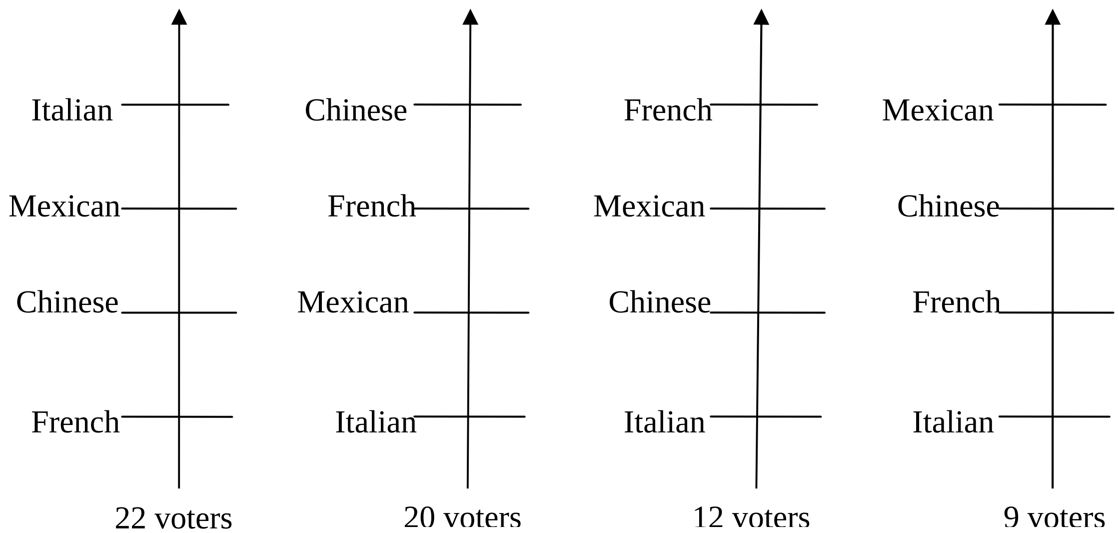
Plurality Method

“A plurality winner is based on first-place rankings only. The winner is the choice that receives the most votes,” (Crisler, Fisher, & Froelich, 2002, p. 8). Most students are familiar with the plurality method; but not familiar with the vocabulary involved with this concept. This method is used in my school, as well as many other schools, for student body elections, homecoming royalty, sports captains, and awards every year. Students are also exposed to this method in local and state political elections. Voters are allowed to vote for only one candidate and the candidate having the most votes wins; whether the winner is a majority winner or not. A majority winner is a candidate that receives over 50% of the total votes.

There are two downfalls with this method. The first is that this method could result in a tie. The second downfall of this method is that when there are more than two candidates; the winner does not have to be a majority. For example, with three candidates the plurality winner could have as little as 33.4% of the votes. Many students do not see a problem with this at first, until you point out that 66.6% of the voters did not want this candidate to win. If there are 10 candidates, the plurality winner could have as little as 10.1% of the votes, with 89.9% of voters not voting for this candidate. Using the plurality method with n candidates, $\{n | n \geq 2 \text{ and } n \in J\}$, the plurality winner only needs to have slightly more than $(100 \div n)$ % of the total vote to win. A downfall of this method is that slightly less than $(100 - 100 \div n)$ % could have placed this candidate last in their list of acceptable winners.

Figure 1.a. represents the preference schedules for 63 members of the math club, with their choices in order from top to bottom. The choice at the top of the preference schedule (closest to the arrow) is the voter's number one choice. The choices are ranked appropriately as you read down the list. The choice ranked second is second from the top, and the choice ranked last is at the bottom of the preference schedule. The members are trying to decide the type of restaurant in which to have their annual meeting. They have decided to rank each restaurant type and evaluate their votes using the six different strategies discussed in this paper.

Figure 1.a Restaurant choices for the 2005 annual meeting



Using the plurality method the winning restaurant is Italian with 22 first place votes. There are 63 total votes, and the Italian restaurant has approximately 34.9% of the first place votes. At first glance, this seems to be a fair representation of the people. Closer inspection shows that the rest of voters, 41 of 63, approximately 65%, placed Italian last. This means that 65% really do not want Italian as their restaurant choice.

Borda Count

The Borda count is named after French mathematician, Jean-Charles de Borda. Borda was unhappy with the plurality method and developed a method of ranking and numbering candidates to find a winner (Crisler, Fisher, & Froelich, 2002). The Borda count has voters rank the candidates in order. Voters can rank in either ascending or descending order, as long as all voters are consistent with each other. To find the Borda winner, points are assigned to each ranking. A voter's last choice for a candidate receives a value of 1, a second to last candidate receives a value of 2, a third to last candidate receives a value of 3, and this pattern continues until all candidates on a voter's schedule have been assigned a point value. For example, if I rank candidates A,B,C, and D in ascending order as A – 2nd choice, B – 1st choice, C – 4th choice, and D – 3rd choice; then candidate C would receive 1 point, D – 2 points, A – 3 points and B – 4 points.

Table 1.a

Candidate	Ranking	Point Value
B	1st	4
A	2nd	3
D	3rd	2
C	4th	1

Every voter's schedule is numbered this way and each candidate's point values are tallied. The candidate with the highest point value wins. This is only one of two ways to implement the Borda count. A similar method is to have each candidate have its respective ranking as its point value and the lowest point value wins; which will result in the same results as the previous point value assignments. An alternate method would be to assign increasing but non-consecutive point values to the choices. An example using 4 candidates could be 1 point for last, 2 points for third, 5 points for second and 10 points for first. Regardless, of the point system that is used, the result should be a choice that the majority of the voters have ranked high on their list.

There can be two downfalls to this method. The first is that the point values can be equal for two or more candidates. The second downfall is that using an alternate point system can result in a different winner. This is a downfall, since a mathematically inclined person could alter the decision in their favor by changing the point system that is used.

Using the preference schedules from figure 1.a. here are 3 different Borda count point systems.

Example 1.b Increasing Consecutive Point System

Last place = 1; 3^{rd} = 2; 2^{nd} = 3; 1^{st} = 4

Chinese = 20(4) + 9(3) + 34(2) = 175

French = 12(4) + 20(3) + 9(2) + 22(1) = 148

Italian = 22(4) + 41(1) = 129

Mexican = 9(4) + 34(3) + 20(2) = 178

Using this system, the highest point value is the winner. The resulting order is: 1^{st} = Mexican, 2^{nd} = Chinese, 3^{rd} = French, and Last = Italian.

Example 1.c Decreasing Consecutive Point System

Last = 4; 3^{rd} = 3; 2^{nd} = 2; 1^{st} = 1

Chinese = 20(1) + 9(2) + 34(3) = 140

French = 12(1) + 20(2) + 9(3) + 22(4) = 167

Italian = 22(1) + 41(4) = 186

Mexican = 9(1) + 34(2) + 20(3) = 137

Using this system, the lowest point value is the winner. The resulting order is: 1^{st} = Mexican, 2^{nd} = Chinese, 3^{rd} = French, and Last = Italian.

Example 1.d Non-Consecutive Point System

Last = 1; 3^{rd} = 2; 2^{nd} = 5; 1^{st} = 10

Chinese = 20(10) + 9(5) + 34(2) = 313

French = 12(10) + 20(5) + 9(2) + 22(1) = 260

Italian = 22(10) + 41(1) = 261

Mexican = 9(10) + 34(5) + 20(2) = 300

Using this system, the highest point value is the winner. The resulting order is: 1st = Chinese, 2nd = Mexican, 3rd = Italian, and Last = French.

With the Borda count, using a consecutive point system, as seen in example 1.b and example 1.c, will result in similar outcomes. Changing to a non-consecutive point system, as seen in example 1.d, will weigh the choices and a different result may be encountered. All three examples 1.b, 1.c, and 1.d conflict with the winner found in example 1.a.

When using this method, students are demonstrating number sense, with their computation and operation skills. To add a higher level of mathematical usage, when using the consecutive point system students can use the sum of an arithmetic series as error analysis. With n candidates and k voters, $\{n \mid n \geq 2 \text{ and } n \in J\}$, and the point system is the set $\{1, 2, \ldots, n\}$ then the sum of all the candidates' Borda count will always equal $S_n \cdot k$; where $S_n = \frac{n(n+1)}{2}$. This total should be checked every time to avoid miscalculation errors.

Runoff Method

Students will find the runoff method to be less time consuming and less complicated than the Borda count. In the runoff method, voters are asked to rank candidates from highest to lowest, highest being their number one choice. This method is much quicker than the Borda count as it immediately narrows all the candidates down to only two options. The two candidates with the most first place votes are kept and all the other candidates are removed. The removed candidates' first place votes go to the remaining candidate that is closest to the top on each voter's preference schedule. Then the first place votes are tallied again, with a majority winner between the two final candidates.

There are downfalls with this method as well. There can be a tie with either the first cut of losers, or when deciding who is the winner between the two final candidates. This method can also result with a remaining candidate having more last place votes than first place; as seen with the Italian restaurant.

Example 1.e.

Using the preference schedules from figure 1.a, count each restaurant's first place votes. Chinese = 20, French = 12, Italian = 22, and Mexican = 9. Since Chinese and Italian are the two highest; they are kept and Mexican and French are removed. French's twelve votes go to Chinese, since it is higher on the preference schedule than Italian. Mexican's nine votes go to Chinese for the same reason. The new first place tallies are; Chinese = 20 + 12 + 9 = 41 and Italian = 22. Thus, the math club will be going to the Chinese restaurant. To check that no votes have been missed, the total votes of the two remaining candidates should equal the total number of voters.

Sequential Runoff Method

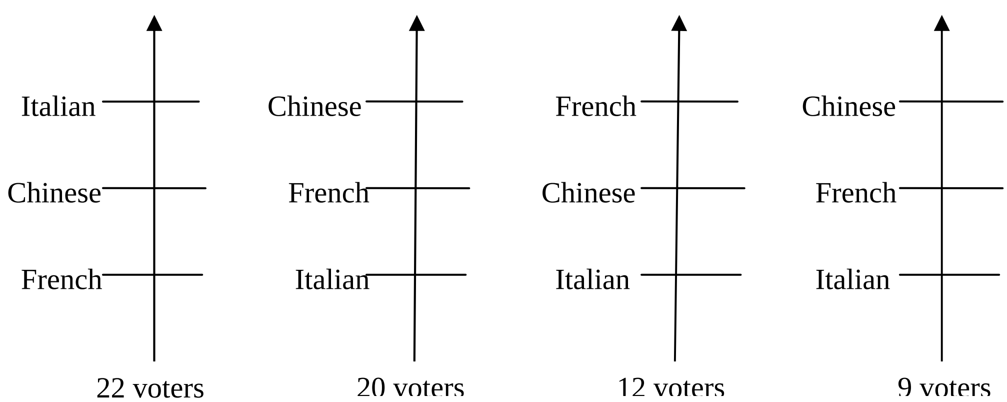
The sequential runoff method is a backward approach of the runoff method. Each candidate's first place votes are tallied and the one candidate with the least is removed from all schedules. Each preference schedule is refigured, with each remaining candidate keeping their respective rankings. Again the candidate with the least first place votes is removed and this method continues until there are only two final candidates, with a majority winner between the two.

This method is very similar to the runoff method in both its downfalls and the final goal of only two candidates. There can be a tie when deciding which candidate to remove as well as a tie between the two final candidates. There is also the possibility that a remaining candidate will have more last place votes than first place votes.

Example 1.f

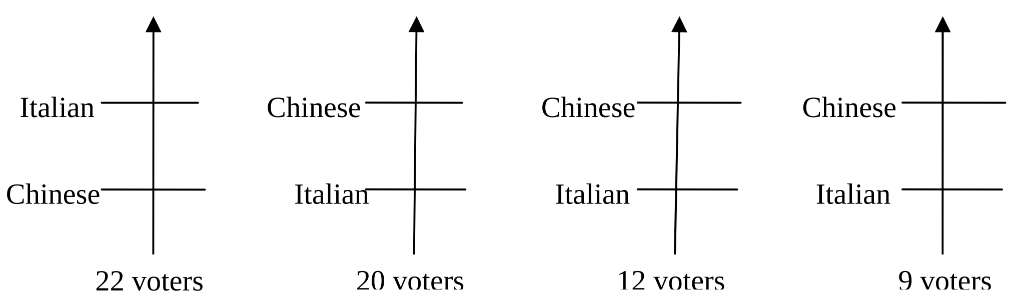
Using figure 1.a, the first place votes are tallied with Chinese = 20, French = 12, Italian = 22, and Mexican = 9. Mexican is removed as a candidate and the preference schedules are then refigured. Every preference schedule with candidates originally ranked lower than the Mexican restaurant will be shifted up to fill the gap. Preference schedules with Mexican in last place are left alone.

Figure 2.a



The first place votes are tallied anew with Italian = 22, Chinese = 29, and French = 12. The candidate with the least first place votes, French, is removed from the preference schedules.

Figure 2.b

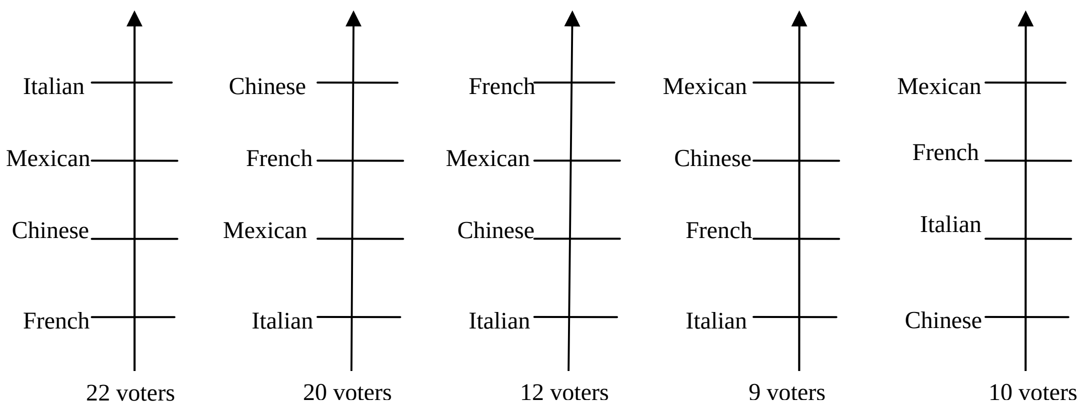


The Chinese restaurant beats the Italian restaurant 41 to 22, the same result as the runoff method.

Example 1.g Runoff and Sequential runoff

The results are not always the same for the runoff and sequential runoff. To show this I've included an extra example comparing the two runoff methods. Using our earlier math club dilemma, ten members came late to the meeting from Spanish club, but were still allowed to vote for their restaurant choice for the annual meeting. By some miracle all ten had the same rankings for the restaurants and their preference schedule was added to the earlier 63 votes in figure 3.c.

Figure 3.c

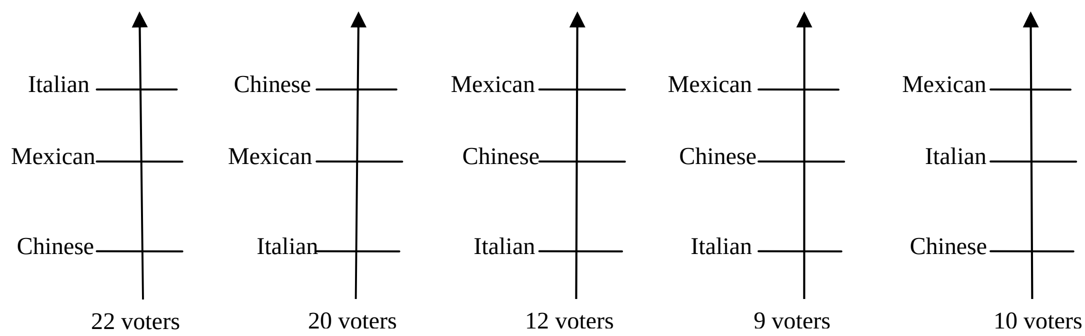


The runoff method has the following first place votes: Chinese = 20, French = 12, Italian = 22, and Mexican = 19. French and Mexican are removed. According to the preference schedule rankings; the twelve votes for the French restaurant go to Chinese, nine votes for the Mexican restaurant go to Chinese, and the remaining 10 votes for the Mexican restaurant go to Italian.

New tallies are Chinese = 41 and Italian = 32. The math club will be dining at the Chinese restaurant.

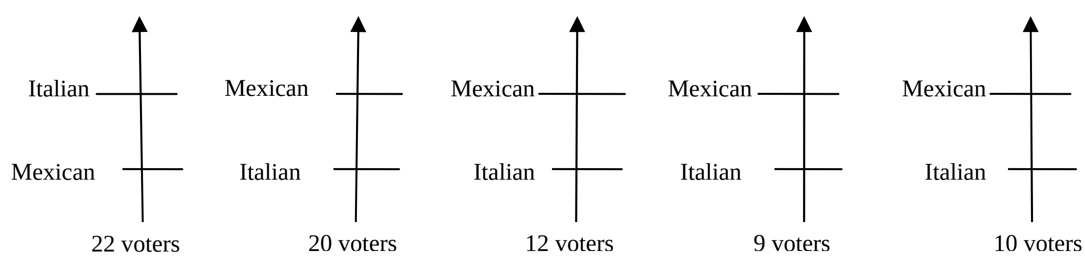
The sequential runoff method has the same original first place votes: Chinese = 20, French = 12, Italian = 22 and Mexican = 19. French is removed and the preference schedules are refigured.

Figure 3.b



The first place votes are recounted with Chinese = 20, Italian = 22, and Mexican = 31. The candidate with the least first place votes is removed, Chinese, and the preference schedules are refigured.

Figure 3.c



The first place votes are tallied again with Italian = 22 and Mexican = 51. According to the sequential runoff method, the math club will be dining at the Mexican restaurant.

Using the runoff method, a top-down approach, the two remaining candidates are Chinese and Italian with Italian winning 41 to 32. Using the sequential runoff method, a bottom-up approach, the two remaining candidates are Italian and Mexican with Mexican winning 51 to 22. This example is important to show to students so they understand that runoff and sequential runoff do not always result in the same winner. The only time that runoff and sequential runoff methods will guarantee the same winner is when it is a three candidate election.

Approval Voting

Unlike the other four methods, this method does not make voters break a tie. If you like candidate A and B equally, then this method allows you to weight each candidate equally. The approval voting method allows all voters to vote for as many candidates as they choose, no ranking is necessary. Voters are encouraged to vote for all choices that would be an acceptable winner for them. In the case of the math club's restaurant dilemma, a voter would vote for all the restaurants that would be acceptable. If a family was voting for their movie rental choice, each member would vote for all the movies that they would like to see. In a political election the voter would vote for all the candidates they would be "happy with" as a leader. The winner is the candidate with the most overall votes.

The downfalls of this method are that a majority winner is not guaranteed, ties are possible, and checking the total vote is not possible. There is no way to know if each voter voted for 1, 2, or more candidates. So when checking the vote tallies there is no way to make sure all votes were counted correctly and no votes are missing. The one guarantee is that with k voters, a candidate's vote total must be a natural number in the interval $[0, k]$.

Example 1.h

When doing example problems with approval voting, assumptions need to be made if there is not real data. Using the preference schedules from figure 1.a, assume that all voters only like to eat at their top two choices. This would result in the following overall tallies: Chinese = 29, French = 32, Italian = 22 and Mexican = 43. The Mexican restaurant wins. Using the same figure 1.a, assume that all voters would approval vote their top 3 choices. This would result in the following overall tallies: Chinese = 63, French = 41, Italian = 22 and Mexican = 63. According to this assumption, all math club members would be content with dining at either the Chinese or the Mexican restaurant. So how do we decide on which restaurant? It's not possible for the math club to dine at both while conducting their annual meeting. Does the math club president choose between the two? Does the treasurer flip a coin? Using figure 1.a again, assume that all voters only approval vote their top choice, then the result would be Chinese = 20, French = 12, Italian = 22, and Mexican = 9. If voters only approval vote their top choice; this result will always be identical to the plurality winner. All of these assumptions are very unrealistic as there is no way to determine the number of candidates that a voter would deem acceptable. Some voters will vote for all the choices, while others will vote for any number of candidates less than that.

Condorcet Winner

Pairwise voting is similar to a single elimination tournament. This method has voters select between two candidates then the winner is paired against another winner. Those two pair off, and the process continues until there is one winner with all other candidates being eliminated. A Condorcet winner is a variation of this pairwise voting method. Voters rank their choices as they did in the Borda, and both runoff methods. Then each candidate is paired against every other candidate and their "wins" and "losses" are tallied. The candidate with the most "wins" is the winner.

The downfalls of this method are that there can be ties between two candidates in deciding the final winner, as well as sometimes violating the transitive property. If you have candidates A, B, and C; with A beating B and B beating C, then naturally we would think that A would beat C when there are paired up. Many times with this method A loses to C and the transitive property is broken, creating a paradox of A > B and B > C, but A < C.

Example 1.g.

Using figure 1.a, start by comparing the Chinese restaurant to the French restaurant on each preference schedule. Chinese is ranked higher than the French restaurant on 51 voters' preference schedules. The next matches would be comparing Chinese to Italian and then to Mexican. Chinese beats Italian 41 to 22 and loses to Mexican 20 to 43. This data is easier to see in a table, called a Condorcet Table.

Table 1.b.

	Chinese	French	Italian	Mexican	Record W-L
Chinese	X	WINS 51-12	WINS 41-22	LOSES 20-43	2-1
French	LOSES 12-51	X	WINS 41-22	WINS 32-31	2-1
Italian	LOSES 22-41	LOSES 22-41	X	LOSES 22-41	0-3
Mexican	WINS 43-20	LOSES 31-32	WINS 41-22	X	2-1

Read the table as rows to columns. Row 1 would read as Chinese wins over French, Chinese wins over Italian, Chinese loses to Mexican, and Chinese's record is 2 wins with 1 loss. From this table we see that there is a three way tie. Chinese, French and Mexican all have a win loss record of 2 wins and 1 loss. When you analyze the table notice that Chinese beat French, French beat Mexican, but Chinese lost to Mexican. This creates a paradox with Chinese > French and French > Mexican, but Chinese < Mexican.

Comparing Methods

Inevitably, students will ask which one of these methods is the “right” method to use. All of these methods have their good points. The plurality method is what we are accustomed to and use for political elections. The Borda count gives every candidate a mathematical value according to all of its rankings. The runoff method assures a winner from the top choices, and the sequential runoff assures the winner will not have the least first place votes. The Condorcet winner pairs each candidate against all others and gets a majority winner between any two candidates, so a Ralph Nadar candidate can’t steal votes. Finally, the approval method allows voters to vote for as many candidates as they choose. All of these are good individual qualities. Unfortunately, there is no method that incorporates all of these together. Let’s compare the winner’s from each method using figure 1.a.

- The Plurality winner is the Italian restaurant with 34.9% of the votes.
- The Borda count winner using both consecutive point systems is the Mexican restaurant.
- The Borda count winner using the non-consecutive point system is the Chinese restaurant.
- The Runoff winner is the Chinese restaurant.
- The Sequential Runoff winner is the Chinese restaurant.
- The Approval Vote winner using the top two choices is the Mexican restaurant.
- The Approval Vote winner using the top three choices has a tie between the Chinese restaurant and the Mexican restaurant.

- The Condorcet winner has a three way tie between Chinese, French, and Italian.

According to our list; the Italian restaurant wins once, the Mexican restaurant wins twice with two ties, and the Chinese restaurant wins three times with two ties. The French restaurant had one win as a tie. Where will the math club eat? Which method is the correct "one"? Which method gives a winner that a majority of voters will feel is an acceptable winner? Once students have mastered the use of these methods, they can focus on these open-ended questions. Making students choose a method and then defend and justify their choice with logical and critical statements will demonstrate several NCTM standards: Reasoning and Proof, Communication, and Connections. The justification of their choice will also demonstrate the 4th level of Bloom's Taxonomy; Analysis.

One way for students to analyze these different methods is by learning and applying Arrow's Theorem and Conditions to them. This theorem was developed by U.S. economist Kenneth Arrow in 1951 (Crisler, Fisher, & Froelich, 2002). The theorem states, "Consider any social welfare function, which always has transitive outcomes for the three or more candidates. Assume that the strictly transitive preference of the voters, where there are at least two of them, satisfy the Universal domain condition. If this procedure satisfies binary independence and Pareto, then it is equivalent to a dictator" (Saari, p. 43, 2001). This theorem can be stated more simply by breaking it down into five conditions that must be met to have a fair group-ranking method.

"Arrow's Conditions

1. Nondictatorship: The preferences of a single individual should not become the group ranking without considering the preferences of the others.

2. Individual Sovereignty: Each individual should be allowed to order the choices in any way and to indicate ties.

3. Unanimity: If every individual prefers one choice to another, then the group-ranking should do the same. (If every voter ranks A>B, then the final ranking should place A>B.

4. Freedom from the Irrelevant Alternatives: The winning choice should still win if one of the other choices is removed. (The choice that is removed is known as the irrelevant alternative.)

5. Uniqueness of the Group Ranking: The method of producing the group ranking should give the same result whenever it is applied to a given set of preferences. The group ranking should also be transitive" (Crisler, Fisher, & Froelich, p. 26-27, 2002).

Arrow eventually proved that there is no method, known or unknown, that can satisfy all five conditions and thus his theorem is an impossibility theorem (Crisler, Fisher, & Froelich, 2002). As students evaluate and analyze the different voting methods they should use Arrow's theorem and conditions in their decision making and justification process. Let's analyze each voting method in regards to Figure 1.a and see where the methods violate Arrow's conditions.

The plurality ranking for our situation is Italian > Chinese > French > Mexican. Where Italian ranks higher than Chinese, Chinese ranks higher than French and French ranks higher than Mexican. The plurality winner will always violate Individual Sovereignty, as voters are never allowed to indicate ties in this method. This particular problem also breaks Freedom from Irrelevant Alternatives, in two cases. If Chinese is removed then the result would be

French (32) > Italian (22) > Mexican (9). If Mexican is removed then the result would be Chinese (29) > Italian (22) > French (12).

The Borda count also breaks the Individual Sovereignty as voters must rank their choices 1st thru last places. The Borda count, when used in a consecutive point system, has the group-ranking Mexican > Chinese > French > Italian. In this point system, number 4, Freedom from Irrelevant Alternatives is violated when Chinese is removed. Using a consecutive 4-3-2 system, the result would be French > Mexican > Italian, as Table 2.a indicates.

Table 2.a.

Restaurant	1st place votes	2nd place votes	3rd place votes	Total Borda Points
Mexican	9	54	0	198
French	32	9	22	199
Italian	22	0	41	170

Using the Borda non-consecutive system from Example 1.d, the original group ranking is Chinese > Mexican > Italian > French. Altering the 10-5-2-1 point assignment to a 10-5-2 point assignment also breaks the Freedom from Irrelevant Alternatives condition when losers are removed. The result is Mexican > Chinese > Italian, when French is removed, see Table 2.b. When Italian is removed, the result is Mexican > Chinese > French, see Table 2.c.

Table 2.b.

Restaurant	1st place votes	2nd place votes	3rd place votes	Total Borda Points
Mexican	21	42	0	420
Chinese	20	21	22	349
Italian	22	0	41	302

Table 2.c.

Restaurant	1st place votes	2nd place votes	3rd place votes	Total Borda Points
Mexican	31	12	20	410
French	12	20	31	282
Chinese	20	31	12	379

The runoff method also breaks both Individual Sovereignty and Freedom from Irrelevant Alternatives. The initial runoff is Italian 22 and Chinese 41. If the French restaurant is removed, then Italian has 22 first place votes, Chinese has 20 first place votes, and Mexican has 21 first place votes. This drops Chinese out of the runoff and the new winner is Mexican 41 to Italian 22. The sequential method from Example 1.f has the exact same results.

The approval method is the first of our methods to not break Individual Sovereignty as all voters are allowed to vote for as many choices as they wish and do not have to break ties. When we assumed that our voters would vote for their top three choices we had a tie. If we break this tie with a flip of the coin then we break condition 5, Uniqueness of the Group Ranking. The coin toss would be random and we can’t guarantee that the same winner would result if we repeated the coin toss again. If we allow the math club president to make the decision then we break condition 1, Nondictatorship. The preference of our president would be the winner, rather than the preference of all the members.

The Condorcet winner breaks Individual Sovereignty since no ties can be indicated and it breaks Uniqueness of the Group Ranking since there is no way to break the tie assuring the same result every time. Similar to the approval voting method ties, the math club could use a random method as a tiebreaker or letting one person choose; which breaks nondictatorship. We also have a paradox which shows that the group ranking is not transitive, since Chinese > French and French > Mexican, but Chinese < Mexican.

As students analyze the methods using Arrow's conditions they should begin to have a better understanding of Arrow's findings. There is no perfect method; the book can't tell them which method is the best nor can I tell them which method is the best. They must decide which method either fits the situation best or satisfies a majority of voters. They must also be able to defend their choice with a reasonable amount of logic and mathematics relevant to the situation.

Lesson Plans

Pre-Class Expectations:

Students were assigned to this class as a required course to replace Algebra II or any higher mathematics course. Students should be an 11^{th} grader, age of 16, or higher. They will have completed and passed a Basic Algebra and an Applied Geometry course before taking this class. Many students will be "algebraically challenged." The class should consist of 15 – 20 students.

Materials Needed:

Students will need writing utensil(s), a notebook, and a desire to learn.

Teacher will need overhead, transparencies, and markers or white board and markers (depending on the resources of the room). Teacher will also need all documents found in the appendix.

Day 1

Start with a very simple discussion of elections. Ask students for examples when they have had the opportunity to have a vote and make a decision. Have a student record these for everyone to see. Accurately account for what the vote was for and how the winner was decided.

You may need to help prod them with suggestions; family, community or local elections. Once the list is finished, discuss if the students felt that is was a fair election. Make them justify why or why not. To add to the list, now ask for examples when their parents or someone they know has had the opportunity to vote or make a decision. You can add to the list to include picking students for scholarships and your own voting experiences. (20 minutes)

Here is a quick and easy mock election activity. Tell them that the next Monday you are going to bring one case of soda to class for them all to drink. Buying 2-liters, 12-packs, or any other amount is not acceptable since you have a coupon at the local store to buy one case of soda for $1.00. To decide what kind of soda to buy you are going to have an election. Give students 5 choices for soda and have them rank them 1 thru 5, with 1 as highest and 5 as lowest. Try to give the students choices that are relevant to them (soda choice of IBC may not be a good candidate if the local store does not offer that product). A simple way to make your decision for 5 soda choices is to monitor your schools vending machines for a week or two to gauge the most popular selections. Collect all votes and show students how to turn each person's vote into a preference schedule. To speed up writing all of the preference schedules you can have students randomly pick a ballot and write the preference schedule on the board or overhead. More than one student can go at a time and if preference schedules are the same the student can indicate this below the schedule. This should keep all students active and engaged during this tedious task. Have all preference schedules visible and have the students write them in their notebooks as this data will be used many times in the next week. (15 minutes) Break students into 4 groups and have them decide who they think the winner should be based on the data. Each group also needs to have a 2nd place, 3rd place, and so on. Each group should be able to explain how they found their ranking and why they used the method they chose. Have each group stand up and explain their choice and keep a record of each group's findings and reasoning. (15 minutes)

Day 2

Define and discuss the plurality winner and majority winner. Did your classroom have a majority winner? A good visual example is having everyone raise their hand that did not vote for the plurality winner. Ask the question: Is there a better way to find the winner? (10 minutes) Explain the runoff method, and list the procedural steps. Have students work thru several examples, finding the plurality winner then the runoff winner. (20 minutes) Put students back into their groups and use the runoff method with their soda preference schedules. Check that each group's work is done correctly and they have the correct runoff winner. (10 minutes) For the last 10 minutes and for homework, have students work on worksheet 1 (Appendix A), individually. Each question represents a different level of Bloom's Taxonomy; Knowledge, Application, and Synthesis in that respective order.

Day 3

Discuss worksheet 1, have several students share their number 3. Ask students to find patterns and try to make generalizations about the examples. What did all the students have the same and what did they have different to create the situation? What methods did they use? (15 minutes) Introduce the Borda count. With our soda choice, assign point values of 5-4-3-2-1 to the first thru last place. (10 minutes) Using the same groups as the previous two days, have them find the Borda point value and check each other's work. While students are working have them answer these two questions: a) if there is a way to check that they have all point values correct b) and how do they know that they didn't make any mathematical errors? Once all groups have found the answer and discussed the two questions, have each group list their results and point values for each candidate. Each group should explain their thoughts on the two questions. If no group realizes that the sum of all the candidates' Borda count will always equal $S_n \cdot k$; where $S_n = \frac{n(n+1)}{2}$ and k = number of voters, point out the pattern and challenge them to come up with the formula. They also need to check that every place has k votes (e.g. there are k first place votes, k 2^{nd} place votes, etc…). (15 minutes) For the remaining time, have the students work thru additional examples, individually. Have students do worksheet 2 (Appendix B) for homework. Using Bloom's Taxonomy, question 1 shows application and question 2 shows Analysis and Evaluation.

Day 4

Discuss worksheet 2. List and compare students' answers to number 2. Point out good examples, so students have an understanding for my expectations on open-ended questions. Ask students to point out explanations that they think are vague or need more explaining, so that students analyze the work of their peers and their own. (10 minutes) Explain the sequential runoff method. (10 minutes) Assign new groups of three. In groups of three apply the sequential runoff method to our original soda election. As students work, have them write which candidate is dropped and give the final two candidate's point total. Is this winner the same as the runoff winner? Have each group share their answers. (15 minutes) For the remaining time, give additional examples and have students find the runoff and sequential runoff. Make sure that there are examples where the winners are the same and where there are different. Ask students to look for a pattern or generalization for when the same winner will be produced with the runoff methods.

Day 5

Start with discussion of when the two runoffs are the same. Did they find any patterns or is it just coincidence when they are the same? (5 minutes) Explain the Condorcet winner and pairwise voting. (10 minutes) Using the same groups as the previous day, have the students make a Condorcet table comparing the five soda choices and determine the winner. Have students answer these questions; a) is there a clear winner? b) are there any paradoxes? c) Are there any cases where a>b and b>c, but a<c? Have groups share and compare Condorcet tables and answers. (15 minutes) For the remaining time, using the example preference schedules from worksheet 1 and worksheet 2, have students find the Condorcet winner, if there is one. Have students answer the same 3 questions as above.

Day 6

List each method's winner for our soda election. Have the students write the answer to this question. "Which candidate do you think should be the winner? Explain why using complete sentences and logic that applies to mathematics from this section." Discuss the different answers, having students analyze and evaluate what they think are acceptable answers and unacceptable answers. (15 minutes) Have the students take a new vote on their soda choice for next week. They can vote for as many soda choices as they want. Basically, they can vote for any soda choice that they would be content with drinking next week. Tally the votes and find the winner. (5 minutes) Introduce approval voting. Does this new method change their answer to the earlier question? How many students would change their answer? Why? Is there a right method? Which method is the best? Have students discuss these questions in their groups from the previous day. Share answers. (10 minutes) Using the remaining time, work on worksheet 3 (Appendix C). Using Bloom's Taxonomy, question 1 shows application and question 2 shows Analysis and Evaluation.

Day 7

Assign new groups of three. Have students do the "How Will you be Evaluated" (Appendix D) activity (Peressini, 1996). This activity uses the stated NCTM Standards, as well as Application, Analysis, and Evaluation from Bloom's Taxonomy. There are additional activities for election theory using all of the discussed voting methods at http://www.colorado.edu/education/DMP/.

Day 8

Discuss Arrow's Theorem, Five Conditions and his resulting conclusion. Using each of our methods from our soda election, decide which conditions each method breaks. Using the groups from day 7, have each group remove a different losing candidate to see if Freedom from Irrelevant Alternatives is broken with each method. The other four conditions are easier to determine when broken without much calculation. Have each group repeat this procedure with the results from worksheet 3. Since our textbook only presents Arrow's Theorem; challenge your higher-level students to find other theorems similar to Arrow's and compare these theorems to Arrow's.

Day 9

Have students use the computer program (Appendix E) to answer questions on worksheet 4 (Appendix F). As a challenge to the more advanced students, have them look over the program and decide if they could alter the program to include finding the runoff and/or sequential runoff winner.

Day 10

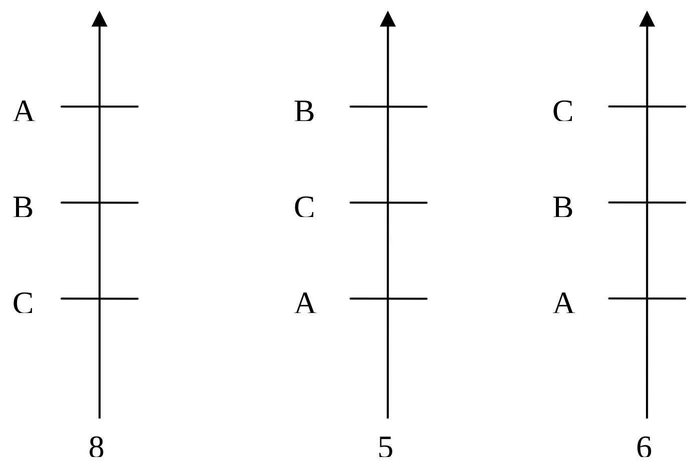
Have students take the Election Theory Test (Appendix G) (McGraw, S.A & Owens, R.W., 2000). I allow any handwritten notes to be used on the test as a reference. The test questions should cover all six levels of Bloom’s Taxonomy. Question 1, shows Application in parts a, b, d, f, g, and h; Analysis and Evaluation are demonstrated in parts c and i; and part d shows Synthesis. Questions 2 and 4 are Application. In answering question 3, students will use Analysis and Evaluation to answer all 5 parts and some students may use synthesis in their justifications. All questions also demonstrate Knowledge and Content, since students need to know the definitions and understand the terminology used in each question and in their answers.

Appendix A.

Worksheet 1. Name ________________________

1. Define a majority winner.

2. Using the following 19 preference schedules, find the plurality winner and the runoff winner. (show and explain all work)



3. Using candidates A, B, C, and D. Create 30 preference schedules where B is the plurality winner, but D is the runoff winner.

Appendix B.

Worksheet 2 Name ________________________

1. Find the plurality, runoff, and Borda count winner for the following preference schedule.

A	B	C	C
D	D	B	D
C	A	D	A
B	C	A	B
20	10	12	15

2. Which candidate do you think should be the winner? Explain why using complete sentences and logic that applies to mathematics from this section.

Appendix C.

Worksheet 3 Name ________________________

1. Use the preference schedule below.

Italian	Chinese	French	Mexican	Mexican
Mexican	French	Mexican	Chinese	French
Chinese	Mexican	Chinese	French	Italian
French	Italian	Italian	Italian	Chinese
22 voters	20 voters	12 voters	9 voters	10 voters

a) Find the plurality winner.

b) Find the runoff winner.

c) Find the Borda count winner and give each candidate’s point value.

d) Find the sequential runoff winner.

e) Find the Condorcet winner.

f) Assume that everyone approval votes their top 2 choices, find the approval winner and give each candidate’s vote total.

2. Which candidate do you think should be the winner? Explain why using complete sentences and logic that applies to mathematics from this section.

Appendix D.

Discrete Mathematics Project
Election Theory Activity

Title

How Will You Be Evaluated? (Dominic D. Peressini)

Goals

1. Students will begin to explore the concept of group-ranking as it relates to election theory.
2. Students will work in small groups in order to arrive at a consensual group-ranking and be able to explain, and justify, how they arrived at this group-ranking to the rest of the class

Abstract

This activity, which is set in the context of having students vote on how their performance will be evaluated for a particular class, focuses on group-ranking. Students are asked to individually determine a rank-ordering of how the five requirements for their class should be weighted. These individual preferences are then tallied, and the students are asked to determine one rank-ordering for the entire class. This activity could be used to introduce Election Theory, and in particular, group-ranking.

Problem Statement

Let students know that as you planned for a class, you thought that it would be nice to come to a group consensus regarding how students' performance in the class would be evaluated. Consequently, you have not yet determined the weighting of the course requirements (assignments). Now as a class, you need to determine the importance of each requirement and weight it accordingly. The first step in determining these weights will be to vote on the rank-order of the assignments.

Instructor Suggestions

1. Set the stage by discussing the "Problem Statement" (see above) with your students.

2. Distribute the "How Will You Be Evaluated?" activity sheet and allow the students to individually read and complete the first part of the activity.
3. When all of the students are finished, have each students write their preference on the board.
4. After all of the preferences have been recorded, have the students form small-groups and determine a class ranking based on the individual rankings.
5. When the small-groups are finished, have a spokesperson for each group share their preference and explain their method and reasoning involved in arriving at their ranking.
6. Discuss the students work as it relates to rank-ordering.

Materials

"How Will You Be Evaluated?" activity sheet, chalk- or dry-erase board.

Time

Introduction of Problem Statement (5 min.), individual work (5 min.), small-group work (20 min.), presentation of small-group work and large-group discussion (15 min.)

Mathematics Concepts

Discrete Mathematics Concepts

Group-Ranking, Plurality Winner, Majority Winner, Borda Method, Runoff Method, Sequential Runoff Method, Condorcet Method (paradox), Arrow's Conditions, Recurrence Relations

Related Mathematics Concepts

Matrices, Sequences and Series

NCTM Standards Addressed

Problem Solving, Communication, Reasoning, Connections (within mathematics and across disciplines), Algebra, Geometry, Discrete Mathematics

How Will You Be Evaluated?

As you look at the syllabus for the Discrete Mathematics Project you will note that the Evaluation section is not complete. More specifically, the weighting of the course requirements (assignments) has not been delineated at this time. In planning the course, we thought that we would come to a group consensus regarding the evaluation. Hence, as a class, we need to determine the importance of each requirement and weight it accordingly.

The first step in determining these weights will be to vote on the rank-order of the assignments. In order to do this, each person will need to list their preference of this rank-order. In particular, you need to rank order the five course requirements (activity development, participation, journals, lesson plans, and reading/home activities) from highest (1) to lowest (5). For example, someone might decide on the following preference:

1. activity development
2. participation
3. journals
4. lesson plans
5. reading/home activities

This person's ordering indicates that he/she feels that "activity development" is the most significant requirement and should carry the most weight in the evaluation of participant achievement for the class. Likewise, this person feels that the reading/home activities should carry the least weight in the evaluation of participant performance.

Please list your preference for the rank-ordering of the DMP class requirements below:

(__________) activity development

(__________) participation

(__________) journals

(__________) lesson plans

(__________) reading/home activities

When everyone has completed their preference, please go to the board and write your individual ranking (since everyone has listed the requirements in the same order--given above--this should require listing only the digits from each persons' preference). List all of the preferences on a sheet of paper.

Your small-group (three or four people) task is to determine a first-, second-, third-, fourth-, and fifth-ranked requirement for the ENTIRE class. You will use all methods discussed in class and then decide which method gives the best group ranking and why. After all of the groups have finished, a spokesperson for each group will write the group's ranking on the board, present the group's decision to the rest of the class, and explain the method (and reasoning) used to reach the decision--it may be helpful to write a description for each step of your method.

Appendix E

Program: Voting

Input “number of preference schedules”, s
Input “number of candidates”, c

Create a matrix A, c x s
Create a matrix B, s x 1

i = 1
j = 1
Disp “Enter each preference schedule in alphabetical order”
While j < = s
while i <= c
Disp “Candidate” i “place value”
Input a_{ij}, a_{ij}
i++
End while
Input “Number of voters on this preference schedule”, b_{j1}
j++
End while

Multiply matrix A*B -> matrix T

j = 1
w = j
While j < s
If $t_{w1}> t_{(j+1)1}$
w = j+1
End if
j++
End while

Disp “the Borda point system used is 1 for first, 2 for 2nd, etc…”
Disp matrix T
Disp “The Borda winner is Candidate” w

Create a matrix P, c x s
i = 1
j = 1
While j < = s
While i <= c
If a_{ij} > 1
p_{ij} = 0

```
else  p_ij = a_ij
End if
i++
End while
j++
End while

Multiply matrix P*B ->T

j = 1
w = j
While j < s
If t_w1< t_(j+1)1
w = j+1
End if
j++
End while

Disp " The plurality winner is candidate" w

Create a matrix V, s x s

i = 1
j = 1
While j < = s
while i <= c
v_ij = 0
If i = j
v_ij = b_i1
End if
i++
End while
j++
End while

Create matrix Z, c x 1
Multiply matrix A*V ->T

i = 1
While i < c
j = 1
y = 0
While y < c
A= 0
B= 0
While j < s
```

If $t_{ij} > t_{(y+1)j}$
Then $b = t_{(y+1)j} + b$
Else $a = t_{ij} + a$
End if
j++
End while
j = 1
If a>b
Then $z_{i1} = 1 + z_{i1}$
Else $z_{(y+1)1} = 1 + z_{(y+1)1}$
End if
y ++
Endwhile
i ++
Endwhile

Disp “The Condorcet wins are as follows”
i=0
While i<=c
Disp “Candidate” i “has” z_{i1} wins”
i++
Endwhile

End Program

Appendix F

Worksheet 4 Name ________________________

Use the computer/calculator program: Voting.

1. Create a 3 candidate voting system, using between 60 and 150 preference schedules, where the plurality winner, Borda winner, and Cordorcet winner are all different. Chart all attempts below and note any patterns that you notice and any conjectures that you find as you go.

2. Create a 6 candidate voting system, using between 500 and 600 preference schedules, where the plurality winner, Borda winner, and Condorcet winner are all different. Apply your patterns and conjectures from the above problem. Chart all attempts and findings below.

Appendix G

Election Theory Test   Name ________________________

Answer all questions and show all work on a separate sheet of paper. Explain all your answers, using complete sentences and using logic and math concepts from this section.

1. An election is being held to choose the president of the Math Anxiety Club. There are four candidates: Aaron, Boris, Carla, and Dawn (A, B, C, and D for short). Each of the 60 members of the club is asked to submit a ballot indicating his or her first, second, third, and fourth choices; ties are not allowed on individual ballots. The 60 ballots submitted are summarized in the following table.

First choice	B	D	C	A	A
Second choice	C	A	D	D	C
Third choice	A	C	A	C	D
Fourth choice	D	B	B	B	B
Number of voters	20	13	8	3	16

a. Find the winner of the election using the plurality method.

b. Find the winner of the election using the runoff method.

c. Could the eight members who most prefer C vote insincerely and change the outcome in part (b) in a way that benefits them? Explain your answer. Assume that they know all the other preferences and that everyone else votes sincerely.

d. Determine the winner of the election using a 4 - 3 - 2 - 1 Borda count.

e. Suppose that you want the Borda winner to be the candidate who finished second using the 4 - 3 - 2 - 1 Borda count in part (d). Find an alternative weighting of the votes in which first place votes receive more weight than second place votes, second place votes receive more weight than third place votes, etc., different from 4 - 3 - 2 - 1, so that your candidate wins.

f. Determine the winner of the election using the sequential runoff method.

g. If possible, determine the Condorcet winner of the election.

h. Suppose that this election is conducted by the approval method and that each voter approves of the first two choices on his or her preference schedule. Determine the approval winner.

i. Considering the information gained in doing parts (a) through (h) of this problem, which candidate do you think is the best choice for president (i.e., which of the voting methods do you think is the most fair in determining a winner)? Explain the reasoning behind your choice.

2. The 20 students on the high school yearbook staff wish to purchase a gift for their advisor. The four items under consideration are a scarf, dinner, theater tickets, and a vase. Assume that they vote sincerely and their preference schedules are as follows:

First choice	scarf	dinner	tickets	vase
Second choice	tickets	vase	dinner	dinner
Third choice	vase	tickets	vase	scarf
Fourth choice	dinner	scarf	scarf	tickets
Total votes	8	4	3	5

a. What choice will the group make if the plurality method is used? Is this choice a good one? Explain.

b. If the two gifts with the fewest first-place votes are eliminated and a run off between the remaining two occurs, which gift will be selected?

c. If the two gifts with the most last-place votes are eliminated and a run off between the remaining two occurs, which gift will be selected?

d. Is there a Condorcet winner? Explain your answer.

3. Consider the following situation. An election is held with four candidates. Each voter submits a complete preference schedule, and the winner is found to be A. Because of a technicality in the voting bylaws, a recount must be performed using the same preference schedules. However, between the time of the original count and the recount, candidate D has withdrawn her candidacy, and the bylaws specify that in this circumstance, her name should be ignored on all preference schedules. In the recount, B is the winner. In parts (a) through (e) below, briefly justify each of your answers.

a. Is this situation possible using the Condorcet method? Explain your answer.

b. Is this situation possible using the plurality voting method? Explain your answer.

c. Is this situation possible using approval voting? Explain your answer.

d. Is this situation possible using a Borda count method? Explain your answer.

e. Which, if any, of Arrow's five conditions does this situation definitely violate? Explain your answer.

4. An election is held among five candidates A, B, C, D, and E using the 5 - 4 - 3 - 2 - 1 Borda count method. There are 24 voters. Suppose that after the ballots are in and the points tallied, A gets 83 points, B gets 79 points, C gets 57 points, and D gets 72 points. How many points did E get and why?

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